Admissible second-order differential subordinations for analytic functions with fixed initial coefficient

Nurshamimi Binti Salleh, Rosihan M. Ali, and V. Ravichandran

Citation: AIP Conference Proceedings **1605**, 655 (2014); doi: 10.1063/1.4887667
View online: http://dx.doi.org/10.1063/1.4887667
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1605?ver=pdfcov
Published by the AIP Publishing

Admissible Second-Order Differential Subordinations for Analytic Functions with Fixed Initial Coefficient

Nurshamimi Binti Salleh[a], Rosihan M. Ali[a] and V. Ravichandran[b]

[a]*School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia*
[b]*Department of Mathematics, University of Delhi, Delhi, 110 007, India*

Abstract. Linear second-order differential subordinations are considered for analytic functions in the unit disk having the same fixed initial coefficient in their series expansion. By showing the subordinations are admissible, this paper describes geometrically the solutions of these subordinations. Connections are also made with earlier known results.

Keywords: Differential subordination, analytic functions, admissible functions, fixed coefficient.
PACS: 02.30.-f

INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions defined in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions p of the form

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots.$$

The subclass of $\mathcal{H}[0,1]$ consisting of normalized univalent functions $f(z) = z + a_2 z^2 + a_2 z^2 + \cdots$ in $\mathbb{U}$ is denoted by S. The second coefficient of functions in S plays an important role in the study of geometric properties of the class S. For this reason, there has been continued interest in the investigations of analytic functions with fixed initial coefficient. These works include those of [1], [2], [3], [4], [5], [6]. The recent work by Ali *et al* .[7] have extended the established theory of differential subordination pioneered by Miller and Mocanu [8] to functions with pre-assigned second coefficient. These results subsequently have been applied by Nagpal and Ravichandran in [9].

In this paper, several new results are obtained for analytic functions with fixed initial coefficient. The methodology used rests on differential subordination by making modifications and improvements to the works developed in [7], [10].

Let $\mathcal{H}_\beta[a,n]$ be the class consisting of analytic functions p of the form

$$p(z) = a + \beta z^n + p_{n+1} z^{n+1} + \cdots$$

with a fixed initial coefficient β in $\mathbb{C}$. Without loss generality, β is assumed to be a nonnegative real number. The following fundamental lemma for functions with fixed initial coefficient will be required.

LEMMA 1. [7, Lemma 2.2, p. 614] Let $q \in Q$ with $q(0) = a$, and $p \in \mathcal{H}_\beta[a,n]$ with $p(z) \top a$. If there exists a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\partial\mathbb{U})$, then

$$z_0 p'(z_0) = m y_0 q'(y_0),$$

and

$$\mathfrak{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \mathfrak{Re}\left(1 + \frac{y_0 q''(y_0)}{q'(y_0)}\right),$$

where $q^{-1}(p(z_0)) = y_0 = e^{i\theta_0}$ and

$$m \geq n + \frac{|q'(0)| - \beta |z_0|^n}{|q'(0)| + \beta |z_0|^n}.$$

Proceedings of the 21st National Symposium on Mathematical Sciences (SKSM21)
AIP Conf. Proc. 1605, 655-660 (2014); doi: 10.1063/1.4887667

Here Q is the class of functions q that are analytic and injective in $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) := \left\{ y \in \partial\mathbb{U} : \lim_{z \to y} q(z) = \infty \right\},$$

and are such that $q'(y) \neq 0$ for $y \in \partial\mathbb{U} \setminus E(q)$.

Let $\psi : \mathbb{C}^3 \to \mathbb{C}$ be analytic in a domain D and let h be univalent in $\mathbb{U}$. If $p \in \mathcal{H}_\beta[a,n]$, $(p(z), zp'(z), z^2 p''(z)) \in D$ when $z \in \mathbb{U}$, and p satisfies the second-order differential subordination

$$\psi\left(p(z), zp'(z), z^2 p''(z)\right) \prec h(z), \tag{1}$$

then p is called a β-*solution* of the differential subordination. A univalent function q is a β-*dominant* of the differential subordination if $p \prec q$ for all p satisfying (1). The class of functions Ψ_β consisting of ψ satisfying (1) is called β-*admissible*.

DEFINITION 1. [7, Definition 3.1, p. 616] (β -Admissibility Condition). Let Ω be a domain in $\mathbb{C}$, $q \in Q$, and $\beta \in \mathbb{R}$ with $\beta \leq |q'(0)|$. The class $\Psi_{n,\beta}(\Omega, q)$ consists of β-*admissible functions* $\psi : \mathbb{C}^3 \to \mathbb{C}$ satisfying the following conditions:

(i) $\psi(r,s,t)$ is continuous in a domain $D \subset \mathbb{C}^3$,

(ii) $(q(0),0,0) \in D$ and $\psi(q(0),0,0) \in \Omega$,

(iii) $\psi(r_0, s_0, t_0) \notin \Omega$ whenever $\psi(r_0, s_0, t_0) \in D$, $r_0 = q(y)$, $s_0 = myq(y)$ and

$$\mathfrak{Re}\left(\frac{t_0}{s_0} + 1\right) \geq m\mathfrak{Re}\left(\frac{yq''(y)}{q'(y)} + 1\right),$$

where $|y| = 1$, $q(y)$ is finite and

$$m \geq n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

The class $\Psi_{1,\beta}(\Omega, q)$ is denoted by $\Psi_\beta(\Omega, q)$.

There are two interesting cases for $q(\mathbb{U})$. First, when $q(\mathbb{U})$ is the disk $\Delta = \{w : |w| < N\}$, and

$$q(z) = N\frac{Nz + a}{N + \bar{a}z}, \qquad z \in \mathbb{U},\ |a| < N,\ N > 0.$$

In this case, set $\Psi_{n,\beta}(\Omega, N, a) \succ \Psi_{n,\beta}(\Omega, q)$ and when $\Omega = \Delta$, denote the class by $\Psi_{n,\beta}(N,a)$. If $a = 0$, then the β-admissibility condition simplifies to

$$\begin{gathered}\psi\left(Ne^{i\theta}, Ke^{i\theta}, L\right) \notin \Omega \quad \textit{whenever} \quad \psi\left(Ne^{i\theta}, Ke^{i\theta}, L\right) \in D, \\ K \geq \left(n + \frac{N-\beta}{N+\beta}\right)N, \quad \textit{and} \quad \mathfrak{Re}\left(Le^{-i\theta}\right) \geq \left(n + \frac{2\beta}{N+\beta}\right)K,\end{gathered} \tag{2}$$

where $\theta \in \mathbb{R}$ and $n \geq 1$. In this particular case for the disk, the following result will be required.

LEMMA 2. [7, Theorem 3.3, p. 619] Let $p \in \mathcal{H}_\beta[a,n]$ with $|a| < N$, $0 < \beta \leq (N^2 - |a|^2)/N$, $N > 0$.

(i) Let $\psi \in \Psi_{n,\beta}(\Omega, N, a)$ with associated domain D. If $(p(z), zp'(z), z^2 p''(z)) \in D$ and

$$\psi\left(p(z), zp'(z), z^2 p''(z)\right) \in \Omega,$$

then $|p(z)| < N$.

(ii) Let $\psi \in \Psi_{n,\beta}(N,a)$ with associated domain D. If $(p(z), zp'(z), z^2 p''(z)) \in D$ and

$$|\psi\left(p(z), zp'(z), z^2p''(z)\right)| < N,$$

then $\left|p(z)\right| < N.$

In the second case where $q(\mathbb{U})$ is the half-plane $\Delta = \{w : \mathfrak{Re}\, w > 0\}$, with

$$q(z) = \frac{a + \bar{a}z}{1 - z}, \qquad z \in \mathbb{U},\ \mathfrak{Re}\, a > 0.$$

Set $\Psi_{n,\beta}(\Omega, a) \rangle\ \Psi_{n,\beta}(\Omega, q)$ and when $\Omega = \Delta$, denote the class by $\Psi_{n,\beta}\{a\}$. If $a = 1$, then the β-admissibility condition reduces to

$$\begin{aligned} &\psi(i\rho, \sigma, \mu + i\upsilon) \notin \Omega \qquad \textit{whenever} \qquad (i\rho, \sigma, \mu + i\upsilon) \in D, \\ &\sigma \leq -\frac{1}{2}\left(n + \frac{2-\beta}{2+\beta}\right)\left(1 + \rho^2\right), \qquad \textit{and} \qquad \sigma + \mu \leq 0, \end{aligned} \tag{3}$$

where $\rho, \sigma, \mu, \upsilon \in \mathbb{R}$ and $n \geq 1$.

The following results relates to the case of the right-half plane.

LEMMA 3. [7, Theorem 3.4, p. 620] Let $p \in \mathcal{H}_\beta[a, n]$ with $\mathfrak{Re}\ a > 0,\ 0 < \beta \leq 2\mathfrak{Re}\ a.$

(i) Let $\psi \in \Psi_{n,\beta}(\Omega, a)$ with associated domain D. If $(p(z), zp'(z), z^2p''(z)) \in D$ and

$$\psi\left(p(z), zp'(z), z^2p''(z)\right) \in \Omega,$$

then $\mathfrak{Re}\ p(z) > 0.$

(ii) Let $\psi \in \Psi_{n,\beta}\{a\}$ with associated domain D. If $(p(z), zp'(z), z^2p''(z)) \in D$ and

$$\mathfrak{Re}\ \psi\left(p(z), zp'(z), z^2p''(z)\right) > 0,$$

then $\mathfrak{Re}\ p(z) > 0.$

MAIN RESULTS

Consider the second-order linear differential subordination

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{4}$$

where $\Omega \subset \mathbb{C}$ and A, B, C, D are complex-valued functions. First let Ω in (4) be the right half-plane.

THEOREM 1. Let n be a positive integer, $0 < \beta \leq 2$, and $A(z) = A \geq 0$. Suppose that the functions $B, C, D : \mathbb{U} \to \mathbb{C}$ satisfy

$$\mathfrak{Re}\ B(z) \geq A \qquad \textit{and}$$

$$\left[\mathfrak{Im}\ C(z)\right]^2 \leq \left(n + \frac{2-\beta}{2+\beta}\right)\left[\mathfrak{Re}\, B(z) - A\right]\left[\left(n + \frac{2-\beta}{2+\beta}\right)\left[\mathfrak{Re}\, B(z) - A\right] - 2\mathfrak{Re}\, D(z)\right].$$

If $p \in \mathcal{H}_\beta[1, n]$ satisfies

$$\mathfrak{Re}\left[Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\right] > 0,$$

then $\mathfrak{Re}\, p(z) > 0.$

PROOF. Letting $\psi(r, s, t; z) = At + B(z)s + C(z)r + D(z)$, the object here is to show that $\psi \in \Psi_{n,\beta}\{1\}$. The function ψ is continuous in the domain $D = \mathbb{C}^3, (1, 0, 0) \in D$ and $\mathfrak{Re}\,\psi(1, 0, 0) = \mathfrak{Re}[C(0) + D(0)] > 0$ so that $\psi(1, 0, 0) \in \Omega.$ To verify the β-admissibility condition (3), consider

$$\psi(i\rho, \sigma, \mu + i\nu; z) = A(\mu + i\nu) + B(z)\sigma + C(z)i\rho + D(z).$$

Then

$\mathfrak{Re}\,\psi(i\rho, \sigma, \mu + i\nu; z)$

$$= A\mu + \left[\Re e\, B(z)\right]\sigma - \left[\Im m\ C(z)\right]\rho + \Re e\, D(z)$$

$$\le -A\sigma + \left[\Re e\, B(z)\right]\sigma - \left[\Im m\ C(z)\right]\rho + \Re e\, D(z)$$

$$\le -\frac{1}{2}\left(n+\frac{2-\beta}{2+\beta}\right)\left(1+\rho^2\right)\left[\Re e\, B(z)-A\right] - \left[\Im m\ C(z)\right]\rho + \Re e\, D(z)$$

$$= -\frac{1}{2}\frac{\left(n+\frac{2-\beta}{2+\beta}\right)^2\left[\Re e\, B(z)-A\right]^2\rho^2}{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]} - \frac{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]\left[\Im m\ C(z)\right]\rho}{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]}$$

$$-\frac{1}{2}\frac{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]\left[\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]-2\Re e\, D(z)\right]}{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]}$$

$$\le -\frac{1}{2}\left\{\frac{\left(n+\frac{2-\beta}{2+\beta}\right)^2\left[\Re e\, B(z)-A\right]^2\rho^2 + 2\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]\left[\Im m\ C(z)\right]\rho + \left[\Im m\ C(z)\right]^2}{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]}\right\}$$

$$= -\frac{1}{2}\frac{\left[\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]+\left[\Im m\ C(z)\right]\right]^2}{\left(n+\frac{2-\beta}{2+\beta}\right)\left[\Re e\, B(z)-A\right]},$$

which is non-positive whenever $\rho,\sigma,\mu,\nu\in\mathbb{R}$ and $n\ge 1$. Hence $\psi\in\Psi_{n,\beta}\{1\}$, and by Lemma 3(ii), we deduce that $\Re e\, p(z)>0$. $\square$

REMARK 1. If $\beta=2$, then Theorem 1 reduces to Theorem 4.1a in [8, p. 188].

COROLLARY 1. Let n be a positive integer, $0<\beta\le 2$, and B,C be complex-valued functions defined in $\mathbb{U}$ with

$$\left|\Im m\ C(z)\right| \le \left(n+\frac{2-\beta}{2+\beta}\right)\Re e\, B(z).$$

If $p\in\mathcal{H}_\beta[1,n]$ satisfies

$$\Re e\left[B(z)zp'(z)+C(z)p(z)\right]>0,$$

then $\Re e\, p(z)>0$.

Next, let Ω in (4) be the disk of radius N centered at the origin, and $A(z)=0$.

THEOREM 2. Let $M>0$, $N>0$, $0<\beta\le N$, and n be a positive integer. Suppose the functions $B,C,D:\mathbb{U}\to\mathbb{C}$ satisfy $B(z)\ne 0$,

(i) $\Re e\left[\frac{C(z)}{B(z)}\right] \ge -\left(n+\frac{N-\beta}{N+\beta}\right)$, *and*

(ii) $\left|\left(n+\frac{N-\beta}{N+\beta}\right)B(z)+C(z)\right| \ge \frac{1}{N}\left[M+\left|D(z)\right|\right]$.

If $p\in\mathcal{H}_\beta[0,n]$ satisfies

$$\left|B(z)zp'(z)+C(z)p(z)+D(z)\right|<M, \tag{5}$$

then $\left|p(z)\right|<N$.

PROOF. Note that $|B(z)zp'(z)+C(z)p(z)+D(z)|<M$ requires that $|D(0)|<M$. Assume that $|p(z)|\not< N$ and let

$$w(z)=B(z)zp'(z)+C(z)p(z)+D(z).$$

According to Lemma 1 with $q(z)=Nz$, there exists a point $z_0\in\mathbb{U}$, $y\in\partial\mathbb{U}$ and an $m\geq n+(N-\beta)/(N+\beta)$ such that $p(z_0)=Ny$ and $z_0p'(z_0)=mNy$. Thus

$$\begin{aligned}|w(z_0)|&=|B(z_0)zp'(z_0)+C(z_0)p(z_0)+D(z_0)|\\&=|B(z_0)mNy+C(z_0)Ny+D(z_0)|\\&=\left|N(mB(z_0)+C(z_0))+\overline{y}D(z_0)\right|\\&\geq N|mB(z_0)+C(z_0)|-|D(z_0)|.\end{aligned}$$

Since $m\geq n+(N-\beta)/(N+\beta)$ and $B(z)\neq 0$, by condition (i) we have

$$0\leq\left(n+\frac{N-\beta}{N+\beta}\right)+\mathfrak{Re}\left[\frac{C(z)}{B(z)}\right]\leq\left|\left(n+\frac{N-\beta}{N+\beta}\right)+\frac{C(z)}{B(z)}\right|\leq\left|m+\frac{C(z)}{B(z)}\right|$$

Therefore ,

$$\left|m+\frac{C(z)}{B(z)}\right|\geq\left|\left(n+\frac{N-\beta}{N+\beta}\right)+\frac{C(z)}{B(z)}\right|,\ and$$

$$|mB(z)+C(z)|\geq\left|\left(n+\frac{N-\beta}{N+\beta}\right)B(z)+C(z)\right|,$$

and condition (ii) yields $|mB(z)+C(z)|\geq\left|\left(n+\frac{N-\beta}{N+\beta}\right)B(z)+C(z)\right|\geq\frac{1}{N}\left(M+|D(z)|\right).$

Thus $\quad |w(z_0)|\geq N|mB(z_0)+C(z_0)|-|D(z_0)|\geq N\frac{1}{N}\left[M+|D(z)|\right]-|D(z_0)|\geq M.$

This contradicts with (5), and hence $|p(z)|<N.$ □

REMARK 2. If $\beta=N$, then Theorem 2 reduces to Theorem 4.1b in [8, p. 190].

The following results, with its proof omitted, extends Theorem 2 in the case $D(z)=0$ and $n=1$.

THEOREM 3. Le t $M>0$, $N>0$, $0<\beta\leq N$, and suppose the functions $B,C:\mathbb{U}\to\mathbb{C}$ satisfy $B(z)\neq 0$, and

$$\frac{M}{N\,|B(z)|}\leq\left|\mathfrak{Im}\left[\frac{C(z)}{B(z)}\right]\right|$$

If $p\in\mathcal{H}_\beta[0,1]$ satisfies

$$|B(z)zp'(z)+C(z)p(z)|<M,\quad then\quad |p(z)|<N.$$

REMARK 3. If $\beta=N$, then Theorem 3 reduces to Theorem 4.1c in [8, p. 192].

Here is the result for the case when $A(z)\neq 0$.

THEOREM 4. Let $M>0$, $N>0$, $0<\beta\leq N$, and n be a positive integer. Suppose the functions $A,B,C,D:\mathbb{U}\to\mathbb{C}$ satisfy $A(z)\neq 0$,

(i) $\quad\mathfrak{Re}\left[\frac{C(z)}{B(z)}\right]\geq-\left(n+\frac{N-\beta}{N+\beta}\right),\quad and$

(ii) $\quad\mathfrak{Re}\left\{\overline{A(z)}\left[B(z)+C(z)\right]\right\}\geq\frac{|A(z)|}{N}\left[M+|D(z)|\right].$

If $p \in \mathcal{H}_\beta[0,n]$ satisfies

$$\left|A(z)z^2 p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\right|<M,$$

then $\left|p(z)\right|<N$.

PROOF. Let $\psi(r,s,t;z)=A(z)t+B(z)s+C(z)r+D(z)$, $\Omega=\{w:|w|<M\}$ and $a=0$. The object is to show that $\psi\in\Psi_{n,\beta}(\Omega,N,0)$. The β-admissibility condition (2) take the form $|\psi_0(z)|\!> \left|\psi\left(Ne^{i\theta},Ke^{i\theta},L;z\right)\right|\geq M$ whenever $m\geq n+(N-\beta)/(N+\beta)$, $K\geq mN, \mathfrak{Re}\left(Le^{-i\theta}\right)\geq\left[n-2\beta/(N+\beta)\right]K$.

Now,

$$\begin{aligned}
\frac{|\psi_0(z)|}{|A(z)|}&=\frac{\left|A(z)L+B(z)Ke^{i\theta}+C(z)Ne^{i\theta}+D(z)\right|}{|A(z)|}\\
&\geq\left|Le^{-i\theta}+\frac{B(z)}{A(z)}K+\frac{C(z)}{A(z)}N\right|-\left|\frac{D(z)}{A(z)}e^{-i\theta}\right|\\
&\geq\mathfrak{Re}\left(Le^{-i\theta}\right)+\mathfrak{Re}\left[\frac{B(z)}{A(z)}\right]mN+\mathfrak{Re}\left[\frac{C(z)}{A(z)}\right]N-\left|\frac{D(z)}{A(z)}\right|\\
&\geq\left(n-\frac{2\beta}{N+\beta}\right)mN+mN\mathfrak{Re}\left[\frac{B(z)}{A(z)}\right]-N\mathfrak{Re}\left[\frac{B(z)}{A(z)}\right]+N\mathfrak{Re}\left[\frac{B(z)}{A(z)}\right]+N\mathfrak{Re}\left[\frac{C(z)}{A(z)}\right]-\left|\frac{D(z)}{A(z)}\right|\\
&=-mN+\left(n+\frac{N-\beta}{N+\beta}\right)N+\mathfrak{Re}\left[\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right]N-\left|\frac{D(z)}{A(z)}\right|\\
&\geq\mathfrak{Re}\left[\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right]N-\left|\frac{D(z)}{A(z)}\right|.
\end{aligned}$$

Thus, $$\left|A(z)\right|^2\frac{|\psi_0(z)|}{\left|A(z)\right|}\geq N\mathfrak{Re}\left[\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right]\left|A(z)\right|^2-\left|\frac{D(z)}{A(z)}\right|\left|A(z)\right|^2,$$

that is, $$\left|A(z)\right||\psi_0(z)|\geq\frac{\left|A(z)\right|}{N}\left[M+\left|D(z)\right|\right]N-\left|A(z)\right|\left|D(z)\right|=\left|A(z)\right|M.$$

This shows that $|\psi_0|\geq M$, and so $\psi\in\Psi_{n,\beta}(\Omega,N,0)$. From Lemma 2(i), we deduce that $\left|p(z)\right|<N$. □

REMARK 4. If $\beta=N$, then Theorem 4 reduces to Theorem 4.1d in [8, p. 193].

ACKNOWLEDGMENTS

The work presented here was supported in part by a research university grant from Universiti Sains Malaysia.

REFERENCES

1. R. M. Ali, N. E. Cho, N. K. Jain and V. Ravichandran, *Filomat* 26:3, 553-561 (2012).
2. R. M. Ali, N. K. Jain and V. Ravichandran, *Appl. Math. Comput.* **281,** 6557-6565 (2012).
3. M. Finkelstein, *Proc. Amer. Math. Soc.* **18,** 412-418 (1967).
4. T. H. Gronwall, *Proc. Nat. Acad. Sci. U. S. A* **6**, 300-302 (1920).
5. K. S. Padmanabhan, *J. Indian Math. Soc. (N.S)* **68**, no. 1-4, 177-189 (2011).
6. D. E. Tepper, *Trans. Amer. Math. Soc*. **150**, 519-528 (1970).
7. R. M. Ali, V. Ravichandran and S. Nagpal, *Bull. Malays.Math.Sci. Soc.* (2) **34**, no.3, 611-629 (2011).
8. S. S. Miller and P. T. Mocanu, *Differential Subordinations*, New York: Marcel Dekker INC., 2000, pp. 187-198.
9. V. Ravichandran and S. Nagpal, *Ann Polon. Math.* **105,** 225-238 (2012).
10. V. Ravichandran, *J. Ineq. in Pure & Applied Math*., (3) **5**, no. 59, (2004).